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FREE VIBRATION BEHAVIOUR OF TAPERED BEAMS WITH NON-LINEAR ELASTIC END ROTATIONAL RESTRAINTS

N. R. NAIDU, G. VENKATESWARA RAO AND K. KANAKA RAJU

Structural Engineering Group, Vikram Sarabhai Space centre, Trivandrum 695022, *India*
E-mail gv-rao@vssc.org

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1. INTRODUCTION

Free vibration and stability behaviour of uniform beams with non-linear elastic end rotational restraint were studied using finite element formulation [1]. For economic and efficient use of material, tapered beams with their near-optimum configuration are effectively used in weight-sensitive space applications of launch vehicle and spacecraft structures. Free vibration characteristics of tapered beams with linear end rotational restraints have been studied by many researchers [2–5]. However, the effect of the non-linear rotational springs on the free vibration behaviour of tapered beams is not readily available in the literature. In continuation of the earlier work [1], the free vibration behaviour of tapered beams with simply supported condition and non-linear end elastic restraints against rotation are studied using the standard finite element formulation. Essentially, the problem addressed presently is a non-linear eigenvalue problem and can be solved, in general, by using a simple iterative technique. If the beam has identical non-linear springs at the ends, the solution can be obtained directly without resorting to any iteration. However, if the rotational springs at the ends have different stiffnesses, the iterative method of solution has to be employed. Numerical results are obtained for both identical and different rotational springs at both ends of the beam and are presented in the form of tables. A brief description of the finite element formulation and the iterative method is given below followed by results and discussion.

2. FINITE ELEMENT FORMULATION

Following reference [1] and generalizing the formulation to consider the variation of geometry along the length of the beam, energy functional π can be written for a tapered beam of length L, moment of inertia I and area of cross-section A and Young's modulus E, with non-linear rotational end restraints at the ends A and B (Figure 1) as

$$\pi = \frac{E}{2}\int_0^L I\, w''^2\,\mathrm{d}x - \frac{\rho\omega^2}{2}\int_0^L A\, w^2\,\mathrm{d}x + \tfrac{1}{2}[M_A\, w']_{x=0} + \tfrac{1}{2}[M_B\, w']_{x=L}, \quad (1)$$

where

$$M_A = \alpha_1\, w' + \beta_1\, w'^3, \quad (2)$$

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$$M_B = \alpha_2 w' + \beta_2 w'^3, \tag{3}$$

$$I = \begin{cases} I_0\left[1 + (\bar{d} - 1)\dfrac{2x}{L}\right]^3, & 0 \leqslant x \leqslant L/2 \\ I_0\left[1 + (\bar{d} - 1)\,2\left(1 - \dfrac{X}{L}\right)\right]^3, & L/2 \leqslant x \leqslant L, \end{cases} \tag{4}$$

$$A = \begin{cases} A_0\left[1 + (\bar{d} - 1)\dfrac{2x}{L}\right], & 0 \leqslant x \leqslant L/2, \\ A_0\left[1 + (\bar{d} - 1)\,2\left(1 - \dfrac{X}{L}\right)\right], & L/2 \leqslant x \leqslant L, \end{cases} \tag{5}$$

$$\bar{d} = d_1/d_0. \tag{6}$$

M_A and M_B are the moments at the ends and α_1, β_1, α_2 and β_2 are the constants which determine the spring behaviour, ρ is the mass density and ω is the circular frequency. It can be interpreted from equations (2) and (3) that α_1, α_2 and β_1, β_2 are the linear and non-linear parts of rotational end restraints respectively. I_0 and A_0 are the moment of inertia and area of cross-section at the ends of the beam and d_1 and d_0 are depths of the beam at the centre and the ends. The depth is considered to vary linearly while the breadth is maintained uniform as shown in Figure 1.

Non-dimensionalizing the length dimensions with respect to L and taking the first variation of equation (1), and equating it to zero we get

$$\delta\pi = \int_0^1 \bar{I} W''\,\delta W''\,\mathrm{d}X - \lambda \int_0^1 \bar{A} W\,\delta W\,\mathrm{d}X$$
$$+ [\bar{\alpha}_1 W'\,\delta W' + 2\bar{\beta}_1 W'^3\,\delta W']_{X=0} + [\bar{\alpha}_2 W'\,\delta W' + 2\bar{\beta}_2 W'^3\,\delta W']_{X=1} = 0. \tag{7}$$

The parameters $\bar{I}$, $\bar{A}$, λ, $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\beta}_1$ and $\bar{\beta}_2$ in equation (7) will be defined later.

Expressing W in terms of the shape functions N_i and nodal displacements $\{W_i\}$ as

$$W = [N_i]\{W_i\}. \tag{8}$$

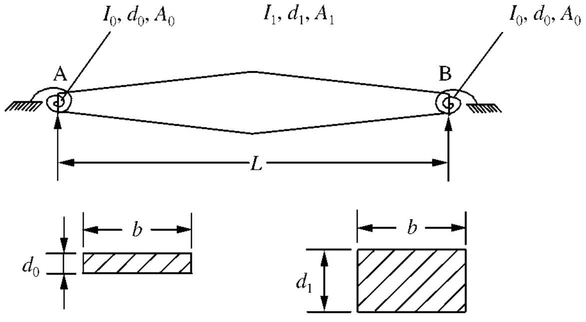

Figure 1. A tapered beam with a non-linear rotational end restraint.

Equation (7) becomes

$$\delta\pi = \int_0^1 \bar{I}\,\{W_i\}^{\mathrm{T}}\,[N_i'']^{\mathrm{T}}\,[N_i'']\,\{\delta W_i\}\,\mathrm{d}X - \lambda \int_0^1 \bar{A}\,\{W_i\}^{\mathrm{T}}\,[N_i]^{\mathrm{T}}\,[N_i]\,\{\delta W_i\}\,\mathrm{d}X$$

$$+ [\bar{\alpha}_1\, W'\,\{\delta W_i\} + 2\bar{\beta}_1\, W'^3\,\{\delta W_i\}]_{X=0} + [\bar{\alpha}_2\, W'\,\{\delta W_i\} + 2\bar{\beta}_2\, W'^3\,\{\delta W_i\}]_{X=1} = 0. \quad (9)$$

After taking transpose, equation (9) can be written as

$$\{\delta W_i\}^{\mathrm{T}} \left[\int_0^1 \bar{I}\,[N_i'']^{\mathrm{T}}\,[N_i'']\,\mathrm{d}X - \lambda \int_0^1 \bar{A}\,[N_i]^{\mathrm{T}}\,[N_i]\,\mathrm{d}X \right.$$

$$\left. + (\bar{\alpha}_1 + 2\,\bar{\beta}_1\, W'^2)_{X=0} + (\bar{\alpha}_2 + 2\,\bar{\beta}_2\, W'^2)_{X=1} \right] \{W_i\} = 0, \quad (10)$$

where λ is the frequency parameter (defined as $\rho\, A_0 \omega^2 L^4/EI_0$). $\bar{I} = I/I_0$ (I is defined by equation (4)), $\bar{A} = A/A_0$ (A is defined by equation (5)), $W = w/L$ and $X = x/L$.

In equation (10) the first term denotes the elastic stiffness matrix and the second term denotes the mass matrix, the third and fourth terms are the equivalent spring stiffness coefficients (where $\bar{\alpha}_1 = \alpha_1 L/EI$, $\bar{\alpha}_2 = \alpha_2 L/EI$, $\bar{\beta}_1 = \beta_1 L/EI$ and $\bar{\beta}_2 = \beta_2 L/EI$) which have to be added to the corresponding diagonal terms of the stiffness matrix.

The matrix equation governing the free vibration after the usual assembly of element matrices can be written as

$$[[K(W')] - \lambda\,[M]]\,\{W\} = 0. \quad (11)$$

Equation (11) can be solved to obtain eigenvalues and eigenvectors using any standard algorithm. It is to be noted that the assembled stiffness matrix contains terms consisting of W' because of the non-linear part of the rotational elastic restraint. However, as has already been mentioned, these non-linear terms appear at two discrete similar points only, i.e., at $X = 0$ and 1. In the case of springs of same stiffness parameters (with same $\bar{\alpha}$ and $\bar{\beta}$) at both ends, any specific W' value can be assumed directly in the stiffness matrix and equation (11) can be solved and no iteration on the solution is necessary. However, if the spring parameters are not identical at the ends, the following iterative method of solution is to be followed to solve equation (11).

3. ITERATIVE METHOD OF SOLUTION

The main steps of the iterative method are

(a) All the W' terms in the stiffness matrix are neglected in equation (11) and the resulting eigenvalue problem is solved to obtain the eigenvalue and eigenvector.
(b) The eigenvector obtained is scaled up so that the value of W' is equal to a prescribed value of W'_1 at the end A. It may be noted here that at this stage the value of W'_2 at the end B also will be the same as W'_1 at the end A.
(c) The spring stiffness values are updated corresponding to the values of W'_1 and W'_2.
(d) These updated spring stiffness values are added to the corresponding diagonal terms of the elastic stiffness matrix to get an updated stiffness matrix.
(e) Using the updated stiffness matrix in equation (11) the eigenvalue and the eigenvector are obtained

(f) Steps (b)–(e) are repeated till the value of W'_2 converges to a specified accuracy (10^{-4} in the present study).
(g) Steps (a)–(f) are repeated for different values of W'_1 at the end A.

4. NUMERICAL RESULTS AND DISCUSSION

By using the finite element formulation described in the previous section, fundamental frequencies of tapered beams with non-linear end elastic rotational restraints have been evaluated. For the symmetric configuration, i.e., identical non-linear springs at the ends, only one-half of the beam is considered and idealized into 10 elements of equal length based on convergence studies. The beam is of rectangular cross-section with linearly varying depth taper as shown in Figure 1. The frequency parameters (λ) are presented for the depth ratio ($\bar{d} = d_1/d_0$) ranging between 1·0 and 1·5 in steps of 0·1. Various values of $\bar{\alpha}(=\bar{\alpha}_1, =\bar{\alpha}_2)$ and $\bar{\beta}(=\bar{\beta}_1, =\bar{\beta}_2)$ and four values of W'_1 $(=W'_2)$ specified at the two ends A and B are considered for the numerical computations. This problem involves only one rotational spring and it is sufficient to use the value of the prescribed rotation W'_1 along with spring constants $\bar{\alpha}$ and $\bar{\beta}$ in the elastic stiffness matrix and solve the eigenvalue problem to get the frequency parameter.

The results are presented in Tables 1–6. It can be observed from these tables that the frequency parameter λ is mainly governed by $\bar{\alpha}$ for small values of $\bar{\beta}$. The effect of non-linearity is predominant when $\bar{\alpha}$ is small and $\bar{\beta}$ and W'_1 are large. As expected, λ is

TABLE 1

Variation of frequency parameter λ with $\bar{\alpha}$, $\bar{\beta}$ and W'_1 for $\bar{d} = \bar{d}_1/d_0 = 1{\cdot}0$

$\bar{\alpha}$	$\bar{\beta}$	W'_1 (deg)			
		0·5	1·0	1·5	2·0
0·0	0·0	97·41	97·41	97·41	97·41
	1·0	97·42	97·43	97·46	97·51
	10·0	97·47	97·65	97·95	98·37
	100·0	98·01	99·80	102·75	106·81
	1000·0	103·34	120·14	145·26	175·41
0·1	0·0	101·32	101·32	101·32	101·32
	1·0	101·33	101·34	101·37	101·41
	10·0	101·38	101·56	101·85	102·26
	100·0	101·91	103·67	106·56	110·55
	1000·0	107·14	123·63	148·31	177·98
1·0	0·0	133·45	133·45	133·45	133·45
	1·0	133·45	133·47	133·49	133·53
	10·0	133·50	133·65	133·90	134·24
	100·0	133·95	135·44	137·90	141·29
	1000·0	138·39	152·46	173·68	199·45
10·0	0·0	298·24	298·24	298·24	298·24
	1·0	298·24	298·24	298·25	298·26
	10·0	298·25	298·30	298·38	298·48
	100·0	298·39	298·85	299·62	300·68
	1000·0	299·77	304·24	311·26	320·29

TABLE 2

Variation of frequency parameter λ with $\bar{\alpha}$, $\bar{\beta}$ and W'_1 for $\bar{d} = d_1/d_0 = 1{\cdot}1$

$\bar{\alpha}$	$\bar{\beta}$	W'_1 (deg)			
		0·5	1·0	1·5	2·0
0·0	0·0	111·35	111·35	111·35	111·35
	1·0	111·35	111·37	111·40	111·44
	10·0	111·41	111·58	111·87	112·28
	100·0	111·93	113·67	116·54	120·48
	1000·0	117·10	133·47	158·07	187·81
0·1	0·0	115·14	115·14	115·14	115·14
	1·0	115·15	115·17	115·20	115·24
	10·0	115·20	115·37	115·66	116·06
	100·0	115·72	117·42	120·24	124·12
	1000·0	120·80	136·88	161·07	190·36
1·0	0·0	146·48	146·48	146·48	146·48
	1·0	146·49	146·50	146·53	146·56
	10·0	146·53	146·68	146·92	147·27
	100·0	146·97	148·43	150·85	154·18
	1000·0	151·33	165·16	186·10	211·69
10·0	0·0	311·51	311·51	311·51	311·51
	1·0	311·52	311·52	311·53	311·54
	10·0	311·53	311·58	311·66	311·77
	100·0	311·67	312·15	312·93	314·02
	1000·0	313·09	317·67	324·88	334·18

TABLE 3

Variation of frequency parameter λ with $\bar{\alpha}$, $\bar{\beta}$ and W'_1 for $\bar{d} = d_1/d_0 = 1{\cdot}2$

$\bar{\alpha}$	$\bar{\beta}$	W'_1 (deg)			
		0·5	1·0	1·5	2·0
0·0	0·0	125·77	125·77	125·77	125·77
	1·0	125·77	125·79	125·82	125·86
	10·0	125·83	126·00	126·28	126·68
	100·0	126·34	128·03	130·82	134·67
	1000·0	131·38	147·36	171·51	200·89
0·1	0·0	129·47	129·47	129·47	129·47
	1·0	129·47	129·49	129·52	129·56
	10·0	129·52	129·69	129·97	130·36
	100·0	130·03	131·69	134·44	138·22
	1000·0	134·98	150·70	174·47	203·42
1·0	0·0	160·12	160·12	160·12	160·12
	1·0	160·12	160·14	160·16	160·19
	10·0	160·17	160·31	160·55	160·89
	100·0	160·60	162·03	164·40	167·68
	1000·0	164·88	178·49	199·20	224·64
10·0	0·0	325·45	325·45	325·45	325·45
	1·0	325·45	325·46	325·46	325·48
	10·0	325·47	325·51	325·60	325·71
	100·0	325·61	326·10	326·90·	328·01
	1000·0	327·06	331·75	339·15	348·70

TABLE 4

Variation of frequency parameter λ with $\bar{\alpha}$, $\bar{\beta}$ and W'_1 for $\bar{d} = d_1/d_0 = 1{\cdot}3$

$\bar{\alpha}$	$\bar{\beta}$	W'_1 (deg)			
		0·5	1·0	1·5	2·0
0·0	0·0	140·65	140·65	140·65	140·65
	1·0	140·66	140·67	140·70	140·74
	10·0	140·71	140·87	141·15	141·54
	100·0	141·21	142·86	145·59	149·35
	1000·0	146·13	161·78	185·52	214·59
0·1	0·0	144·26	144·26	144·26	144·26
	1·0	144·27	144·29	144·31	144·35
	10·0	144·32	144·48	144·76	145·14
	100·0	144·81	146·44	149·12	152·83
	1000·0	149·65	165·06	188·44	217·10
1·0	0·0	174·31	174·31	174·31	174·31
	1·0	174·31	174·33	174·35	174·38
	10·0	174·36	174·50	174·73	175·06
	100·0	174·78	176·19	178·53	181·75
	1000·0	178·99	192·42	212·91	238·22
10·0	0·0	339·99	339·99	339·99	339·99
	1·0	339·99	339·99	340·00	340·01
	10·0	340·00	340·05	340·14	340·25
	100·0	340·15	340·65	341·47	342·61
	1000·0	341·63	346·43	354·00	363·81

TABLE 5

Variation of frequency parameter λ with $\bar{\alpha}$, $\bar{\beta}$ and W'_1 for $\bar{d} = d_1/d_0 = 1{\cdot}4$

$\bar{\alpha}$	$\bar{\beta}$	W'_1 (deg)			
		0·5	1·0	1·5	2·0
0·0	0·0	155·98	155·98	155·98	155·98
	1·0	155·99	156·00	156·03	156·07
	10·0	156·04	156·20	156·47	156·85
	100·0	156·52	158·14	160·81	164·50
	1000·0	161·34	176·70	200·08	228·86
0·1	0·0	159·52	159·52	159·52	159·52
	1·0	159·52	159·54	159·57	159·60
	10·0	159·57	159·73	160·00	160·37
	100·0	160·05	161·64	164·28	167·91
	1000·0	164·80	179·92	202·97	231·36
1·0	0·0	189·02	189·02	189·02	189·02
	1·0	189·03	189·04	189·06	189·10
	10·0	189·07	189·21	189·44	189·77
	100·0	189·49	190·88	193·18	196·36
	1000·0	193·64	206·89	227·20	252·40
10·0	0·0	355·08	355·08	355·08	355·08
	1·0	355·08	355·09	355·10	355·11
	10·0	355·10	355·15	355·23	355·35
	100·0	355·25	355·76	356·59	357·75
	1000·0	356·76	361·66	369·40	379·45

TABLE 6

Variation of frequency parameter λ with $\bar{\alpha}$, $\bar{\beta}$ and W'_1 for $\bar{d} = d_1/d_0 = 1{\cdot}5$

$\bar{\alpha}$	$\bar{\beta}$	W'_1 (deg)			
		0·5	1·0	1·5	2·0
0·0	0·0	171·74	171·74	171·74	171·74
	1·0	171·75	171·76	171·79	171·83
	10·0	171·79	171·95	172·22	172·59
	100·0	172·27	173·86	176·48	180·10
	1000·0	177·00	192·09	215·16	243·68
0·1	0·0	175·21	175·21	175·21	175·21
	1·0	175·21	175·23	175·26	175·29
	10·0	175·26	175·42	175·68	176·05
	100·0	175·73	177·30	179·88	183·45
	1000·0	180·39	195·26	218·01	246·16
1·0	0·0	204·23	204·23	204·23	204·23
	1·0	204·24	204·25	204·28	204·31
	10·0	204·28	204·42	204·65	204·97
	100·0	204·69	206·07	208·34	211·47
	1000·0	208·79	221·89	242·03	267·13
10·0	0·0	370·70	370·70	370·70	370·70
	1·0	370·70	370·71	370·71	370·73
	10·0	370·72	370·77	370·85	370·97
	100·0	370·87	371·38	372·24	373·42
	1000·0	372·41	377·40	385·31	395·58

TABLE 7

Variation of frequency parameter λ and W'_2 with $\bar{\alpha}_1$, $\bar{\beta}_1$, $\bar{\alpha}_2$, $\bar{\beta}_2$, and W'_1 for $\bar{d} = d_1/d_0 = 1{\cdot}5$

$\bar{\alpha}_1$	$\bar{\beta}_1$	W'_1 (deg)	$\bar{\alpha}_2$	$\bar{\beta}_2$	λ	W'_2 (deg)
0·1	1·0	2·0	0·1	1·0	175·29	2·0
0·1	1·0	2·0	0·1	10·0	175·32	1·998
0·1	1·0	2·0	1·0	10·0	189·42	1·801
0·1	10·0	2·0	0·1	1·0	175·32	2·010
0·1	10·0	2·0	0·1	10·0	176·05	2·0
0·1	10·0	2·0	1·0	10·0	189·82	1·806
1·0	10·0	2·0	0·1	1·0	189·44	2·238
1·0	10·0	2·0	0·1	10·0	189·93	2·229
1·0	10·0	2·0	1·0	10·0	204·97	2·0

found to increase with increase in depth parameter. For smaller values of $\bar{\alpha}$, λ increases with increasing $\bar{d}$ and the increase is about 10–14% with each 0·1 step increase of $\bar{d}$, whereas for higher values of $\bar{\alpha}$, λ increases by about 4.4%. The percentage increase is more (14%) when $\bar{d}$ values are small (1·0–1·1, 1·1–1·2) and is about 10% when $\bar{d}$ values are large (1·4–1·5).

For unsymmetrical configuration, i.e., the two end rotational springs having different spring constants, the full beam is idealized into 20 equal finite elements (full beam is considered due to unsymmetric restraints) and an iterative solution scheme, explained in the previous section is followed. Input to this case is $\bar{\alpha}_1$, $\bar{\beta}_1$, W'_1, $\bar{\alpha}_2$ and $\bar{\beta}_2$. The value of λ and W'_2

of the tapered beam for a depth ratio of 1·5 are given in Table 7. The results converged to a specified accuracy of 10^{-4} in two iterations. As a check on the iterative scheme some typical cases, when $\bar{\alpha}_1 = \bar{\alpha}_2$ and $\bar{\beta}_1 = \bar{\beta}_2$ with W'_1 have been solved and the solution, as expected converged in one iteration with the value of W'_2 obtained coinciding with W'_1 and the λ value coinciding with the value given in Table 6. It can be seen from Table 7 that for a given W'_1, the value of W'_2 is either greater or less than W'_1 depending upon whether the equivalent spring stiffness at the end B is smaller or larger than that at the end A.

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APPENDIX A: NOMENCLATURE

A_0	area of cross-section at the ends of the beam
A	area of cross-section at any section of the beam
b	breadth of the beam
d_0	depth at the ends of the beam
d_1	depth at the centre of the beam
$\bar{d}$	d_1/d_0
E	Young's modulus
I	moment of inertia at any section of the beam
$[K]$	elastic stiffness matrix
L	length of the beam
$[M]$	mass matrix
M_A, M_B	moments at the ends of the beam
$[N_i]$	shape functions
w	displacement
W	w/L
x	axial co-ordinate
X	x/L
α_1, α_2	spring stiffness coefficients
$\bar{\alpha}_1$	$= \alpha_1 L/(EI)$
$\bar{\alpha}_2$	$= \alpha_2 L/(EI)$
β_1, β_2	spring stiffness coefficients
$\bar{\beta}_1$	$= \beta_1 L/(EI)$
$\bar{\beta}_2$	$= \beta_2 L/EI)$
ρ	mass density
π	energy functional
λ	frequency parameter $(\rho A_0 \omega^2 L^4)/(EI_0)$

Superscripts

,	differentiation with respect to x
T	transpose